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A note on the stability of convection in a vertical slot

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The study of natural convection in a rectangular cavity whose side walls are maintained at different fixed temperatures can be regarded as one of the classical problems of thermal convection (see Batchelor 1954; Elder 1965 and Gill 1966). One aspect of this problem is the question of stability of the laminar flow solution for given values of the three governing parameters, the Rayleigh number $A = \gamma g\Delta TL^3/\kappa\nu$, the aspect ratio $h = H/L$ and the Prandtl number $\sigma = \nu/\kappa$. Here H is the height and L the width of the cavity, T is the temperature difference between the two walls and g the acceleration due to gravity. The fluid filling the container has coefficient of expansion γ, thermal diffusivity κ and kinematic viscosity ν. Instead of studying the stability of the exact laminar flow solution, which is not a parallel flow, the stability of the following solution of the governing equations is examined instead:

$$\left.\begin{aligned} T &= T_0+\Delta T\left[\frac{1}{2}\frac{z}{H}+\Theta\left(\frac{x}{L}\right)\right], \\ W &= \gamma g\Delta TL^2\nu^{-1}W(x/L), \end{aligned}\right\} \qquad (1)$$

where $$\Theta - 2m^2 iW = \tfrac{1}{2}\sinh\{(1+i)mx/L\}/\sinh\{(1+i)m/2\}. \qquad (2)$$

In the above expressions (x, z) are rectangular co-ordinates such that the z-axis points vertically upwards and the side walls of the cavity are at $x = \pm\frac{1}{2}L$. w is the vertical velocity, T_0 is a constant, and m is given by

$$m^4 = A/8h.$$

On the side walls, $x = \pm\frac{1}{2}L$, $w = 0$, and $T = T_0+\Delta T\{\frac{1}{2}(z/H)\pm\frac{1}{2}\}$. Clearly this is not the solution for a rectangular cavity with isothermal side walls, but it has been found experimentally (Elder 1965) that this solution gives a reasonable fit to observations made near the centreline ($x = 0$) of slots with large aspect ratio h.

The stability to *stationary* disturbances of the system defined by (1) and (2) has been examined by Vest & Arpaci (1969).† In the limit $m \to 0$, (i.e. $h \to \infty$), (2) reduces to

$$\Theta = x, \quad W = \tfrac{1}{6}(\tfrac{1}{4}x - x^3),$$

and the classical result for convection between infinite parallel vertical plates is recovered. The stability of this system has been studied by Gershuni (1953),

† *Note added in proof:* It has been drawn to our attention that Vest & Arpaci ignored the effect of the vertical temperature gradient on the stability, and so their results may not be meaningful.

Rudakov (1967) and others. In particular, Rudakov found that this system is most unstable to *stationary* disturbances if the Prandtl number is less than 10. Vest & Arpaci (1969) found that the critical value of $\sigma^{-1}A$ for stationary disturbances is practically independent of σ in the range $0 \leqslant \sigma \leqslant 1000$. In the limit $m \to \infty$ (i.e. $A \to \infty$), (2) becomes, in the neighbourhood of $x = -\frac{1}{2}L$,

$$\Theta - 2m^2 iW = \tfrac{1}{2}\exp\{-(1+i)m[(x/L)+\tfrac{1}{2}]\}, \qquad (3)$$

and the significant effects are confined to a thin boundary layer. The stability of the system defined by (1) and (3) has been examined by Gill & Davey (1969), it being found that the system is unstable to *travelling* disturbances. In particular, they found that the calculation of the neutral stability curve was relatively simple in the limit of infinite Prandtl number, σ, it being only necessary to solve a fourth-order equation with real coefficients instead of a sixth-order equation with complex coefficients.

The present note is concerned with the study of this limiting problem for finite values of m. More details than are given here can be found in a dissertation (Kirkham 1969). The limit may be defined as follows:

$$\sigma^{-\frac{1}{2}}A = S = \text{constant},\ m = \text{constant (and therefore } \sigma^{-\frac{1}{2}}h = \text{constant).}$$

The analytical procedure for calculating the neutral stability curve is outlined in §5 of Gill & Davey (1969). The results for critical conditions (onset of instability) are given in the table below. For large m the results agree well with those of Gill & Davey (1969, table 1), whose result in terms of the present notation is

$$S = 15{\cdot}0 \times 4m^3, \quad \alpha = 0{\cdot}42 \times m, \quad c = 0{\cdot}51/4m^2.$$

m	S	α	c
40	$3{\cdot}8 \times 10^6$	17·0	$7{\cdot}9 \times 10^{-5}$
10	$6{\cdot}1 \times 10^4$	4·6	$1{\cdot}2 \times 10^{-3}$
8	$3{\cdot}3 \times 10^4$	4·0	$1{\cdot}8 \times 10^{-3}$
6	$1{\cdot}7 \times 10^4$	3·6	$3{\cdot}0 \times 10^{-3}$
4	$1{\cdot}0 \times 10^4$	3·0	$5{\cdot}6 \times 10^{-3}$
1	$9{\cdot}4 \times 10^3$	2·5	$8{\cdot}5 \times 10^{-3}$
0	$9{\cdot}4 \times 10^3$	2·5	$8{\cdot}5 \times 10^{-3}$

TABLE 1. Critical values of S with corresponding values of the non-dimensional wave-number α and non-dimensional wave speed c. The dimensional wave-number and wave-speed are $L^{-1}\alpha$ and $\gamma g\,\Delta T\,L^2\nu^{-1}c$, respectively.

The neutral waves have phase speed greater than the maximum vertical velocity for all values of m. The excess velocity is relatively small for small values of m—when the maximum vertical velocity is (for $m = 0$) $8{\cdot}0 \times 10^{-3}$. The growth rates of unstable disturbances of this type are very small, being of order $\sigma^{-\frac{1}{2}}$. For instance, for $m = 0$, if S is twice the critical value, then the fractional increase in amplitude in the time taken to move a distance L at the maximum vertical velocity is only $1{\cdot}0 \times \sigma^{-\frac{1}{2}}$. The most interesting conclusion of this study is that, at large Prandtl numbers, natural convection between infinite vertical parallel plates (the case $m = 0$) is unstable to travelling disturbances ($A \propto \sigma^{\frac{1}{2}}$) before it is unstable to stationary disturbances ($A \propto \sigma$). This is the reverse of the situation for smaller Prandtl numbers (see Rudakov 1967). The present results, along with those of Vest & Arpaci (1969), mean that the stability diagram for a vertical slot presented by Gill & Davey (1969, figure 17) needs considerable modification, at least for large Prandtl numbers. The revised diagram, for $\sigma = 1000$, is shown in the figure. It is assumed that 1000 is a large enough value of the Prandtl number of the present results to be applicable.

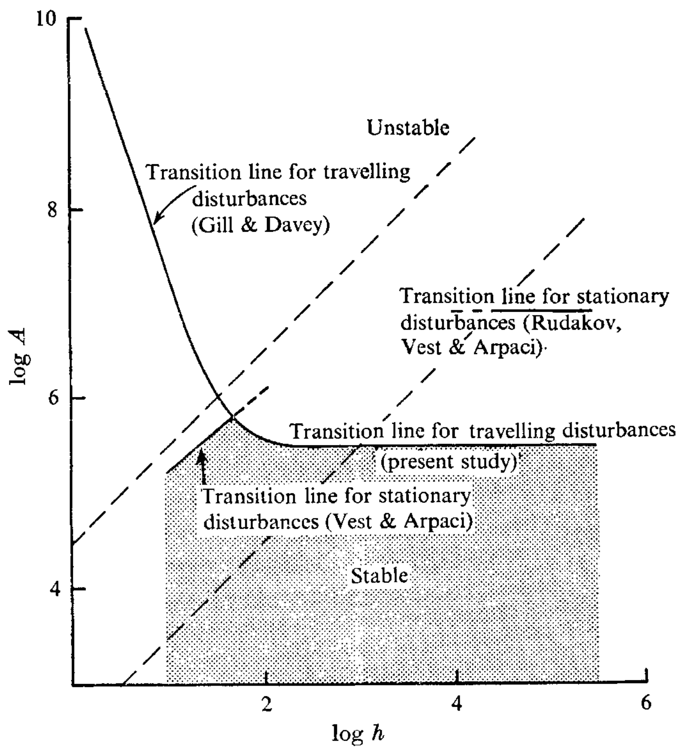

FIGURE 1. Stability characteristics of convection in a vertical slot as a function of Rayleigh number A and aspect ratio h for a Prandtl number of 1000. The laminar solution is conduction dominated in the lower right of the diagram and convection dominated in the upper left. In the region between the thin dashed lines it is neither convection nor conduction dominated.†

† *Note added in proof:* The appropriateness of the sloping transition line of Vest & Arpaci is in doubt (see previous footnote).

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